

Lower and upper bounds for H -eigenvalues of even order real symmetric tensors

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Abstract

In this article, we define new classes of tensors called double \overline{B} -tensors, quasi-double \overline{B} -tensors and establish some of their properties. Using these properties, we construct new regions viz., double \overline{B} -intervals and quasi-double \overline{B} -intervals, which contain all the H -eigenvalues of real even order symmetric tensors. We prove that the double \overline{B} -intervals is contained in the quasi-double \overline{B} -intervals and quasi-double \overline{B} -intervals provide supplement information on the Brauer-type eigenvalues inclusion set of tensors. These are analogous to the double \overline{B} -intervals of matrices established by J. M. Peña [On an alternative to Gerschgorin circles and ovals of Cassini, Numer. Math. 95 (2003), no. 2, 337-345.]

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1 Introduction

A tensor can be regarded as a higher-order generalization of a matrix, which takes the form

$$\mathcal{A} = (a_{i_1 \dots i_m}), \quad a_{i_1 \dots i_m} \in \mathbb{R}, \quad i_j \in [n] := \{1, \dots, n\}, \quad j \in [m]. \quad (1)$$

Such a multidimensional array is called an m -order n -dimensional real tensor and the set of all m -order n -dimensional real tensors is denoted by $\mathcal{T}(\mathbb{R}^n, m)$. A tensor \mathcal{A} is called *symmetric* if its entries $a_{i_1 \dots i_m}$ are invariant under any permutation of their indices $\{i_1, \dots, i_m\}$.

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Let $\mathbb{R}^n(\mathbb{C}^n)$ denote the n -dimensional real (complex) vector space. Vectors are denoted by lower case letters (x, y, \dots) , matrices by upper case letters (A, B, \dots) and tensors by calligraphic upper case letters $(\mathcal{A}, \mathcal{B}, \dots)$. The i^{th} entry of a vector x is denoted by x_i , the $(i, j)^{\text{th}}$ entry of a matrix A is denoted by a_{ij} and the $(i_1, \dots, i_m)^{\text{th}}$ entry of a tensor \mathcal{A} is denoted by $a_{i_1 \dots i_m}$.

Given a vector $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n$, we define $\mathcal{A}x^{m-1}$ to be a vector in \mathbb{C}^n whose i th coordinate is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}. \quad (2)$$

For $x \in \mathbb{C}^n$ and a natural number k , the vector $x^{[k]}$ is the Hadamard power of x , i.e. $x_i^{[k]} = x_i^k$ for all i .

Definition 1.1. [8] Let $\mathcal{A} \in \mathcal{T}(\mathbb{R}^n, m)$. If there is a nonzero vector $x \in \mathbb{C}^n$ and a number $\lambda \in \mathbb{C}$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$$

, then λ is called an *eigenvalue* of the tensor \mathcal{A} and x an *eigenvector* of \mathcal{A} associated with λ . Furthermore, we say λ is an *H-eigenvalue* with the corresponding *H-eigenvector* of \mathcal{A} if they are real.

A symmetric tensor $\mathcal{A} \in \mathcal{T}(\mathbb{R}^n, m)$ is said to be *positive semidefinite* if for any vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m := \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \cdots x_{i_m} \geq 0$; \mathcal{A} is *positive definite* if for any nonzero vector $x \in \mathbb{R}^n$, $\mathcal{A}x^m > 0$. From the definition it is clear that, if m is odd, then there is no nontrivial positive semidefinite tensor.

Positive definite homogenous polynomials and positive semidefinite polynomials (non-negative polynomials) are important in the field of dynamical systems, optimization, etc. With each homogenous polynomial we can associate a symmetric tensor. Checking the positive (semi)definiteness of a homogenous polynomial is equivalent to checking the positive (semi)definiteness of the symmetric tensor associated with it. For details about the applications we refer to [6], [8] and the references therein.

Qi characterized the positive definite and positive semidefinite tensors in terms of their *H-eigenvalues*.

Theorem 1.1. [8, Theorem 5] Let \mathcal{A} be an m -order n -dimensional symmetric tensor such that m is an even integer. Then \mathcal{A} is positive definite (positive semidefinite) if and only if all its *H-eigenvalues* are positive (nonnegative).

Thus, from the above theorem, the location of the *H-eigenvalues* of an even order symmetric tensor is useful in checking the positive definiteness (positive semidefiniteness) of tensors. The main purpose of this article is to give upper and lower bounds for the *H-eigenvalues* of even order symmetric tensors. Next we recall a couple of known results in this direction.

For each $i \in [n]$, denote

$$r_i(\mathcal{A}) = \sum \{|a_{ii_2 \dots i_m}| : i_j \in [n], j = 2, \dots, m, (i_2, \dots, i_m) \neq (i, \dots, i)\}. \quad (3)$$

Qi et al. [8, 9] asserted that the eigenvalues of a tensor \mathcal{A} have a similar statement as the Gerschgorin circle of the eigenvalues of matrices. One can rewrite the result for H -eigenvalues of a real tensor as follows.

Theorem 1.2. [9, Theorem 2] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ and λ be an H -eigenvalue of \mathcal{A} . Then,*

$$\lambda \in \Gamma(\mathcal{A}) = \bigcup_{i=1}^n \Gamma_i(\mathcal{A}), \quad (4)$$

where $\Gamma_i(\mathcal{A}) = [a_{i \dots i} - r_i(\mathcal{A}), a_{i \dots i} + r_i(\mathcal{A})]$.

We call the interval $\Gamma(\mathcal{A})$ in (4) the Gerschgorin eigenvalues inclusion set of tensors. Li et al. established a Brauer-type eigenvalues inclusion set for an arbitrary complex tensor [4, Theorem 2.1] and showed that the Brauer-type eigenvalues inclusion set is contained in the Gerschgorin eigenvalues inclusion set [4, Theorem 2.3]. We can restate the Brauer-type eigenvalues inclusion theorem for H -eigenvalues of a real tensor as follows:

Theorem 1.3. [4, Theorem 2.1] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ and λ be an H -eigenvalue of \mathcal{A} . Then,*

$$\lambda \in \Omega(\mathcal{A}) = \bigcup_{\substack{i,j=1 \\ i \neq j}}^n \Omega_{ij}(\mathcal{A}), \quad (5)$$

where

$$\Omega_{ij}(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \dots i}|(|z - a_{j \dots j}| - r_j^i(\mathcal{A})) \leq r_i(\mathcal{A})|a_{ji \dots i}|\},$$

and

$$r_j^i(\mathcal{A}) = r_j(\mathcal{A}) - |a_{ji \dots i}| = \sum_{\substack{j_2, \dots, j_m=1 \\ (j_2, \dots, j_m) \neq (i, \dots, i) \\ (j_2, \dots, j_m) \neq (j, \dots, j)}}^n |a_{jj_2 \dots j_m}|.$$

In this article, in Section 2, we collect some definitions, known results and correct minor mistakes in couple of results proved in [3]. In Section 3, we define two new classes tensors viz., double \overline{B} -tensors, quasi-double \overline{B} -tensors and establish their properties. Using this properties, in Section 4, we construct two new regions called double \overline{B} -intervals and quasi-double \overline{B} -intervals, containing all the H -eigenvalues of even order symmetric tensors. These regions have a nature similar to the Brauer-type eigenvalues inclusion set $\Omega(\mathcal{A})$ stated in (5). We prove that quasi-double \overline{B} -intervals is smaller than double \overline{B} -intervals and there is no inclusion relation between quasi-double \overline{B} -intervals and the Brauer-type eigenvalues inclusion set. Hence, it is nature to construct an intervals $\Upsilon(\mathcal{A})$, narrower than both quasi-double \overline{B} -intervals and Brauer-type eigenvalues inclusion set $\Omega(\mathcal{A})$, by intersecting the two. These comparison results are done in Section 5. In Section 6, we draw some concluding remarks.

2 Notation, Definitions and Preliminary results

The purpose of this section is twofold - to collect known definitions and results; and to correct minor mistakes in couple of results in [3].

The m -order n -dimensional *identity tensor*, denoted by \mathcal{I} , is the tensor with entries

$$a_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Chen et al. [1] defined the following k th row tensor dealing with some problems of circulant tensors.

Definition 2.1. [1] Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$. Then, $\mathcal{A}_k = (a_{i_1 \dots i_{m-1}}^{(k)}) \in \mathcal{T}(\mathbb{R}^n, m-1)$ is called the k th row tensor if $a_{i_1 \dots i_{m-1}}^{(k)} = a_{ki_1 \dots i_{m-1}}$, where $k, i_1, \dots, i_{m-1} \in [n]$.

Let $\text{sign}(x)$ be sign function, that is,

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

For a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ and $i \in [n]$, define

$$\beta_i(\mathcal{A}) = \max\{0, a_{ii_2 \dots i_m} : (i_2, \dots, i_m) \neq (i, \dots, i)\}, \quad (6)$$

$$\gamma_i(\mathcal{A}) = \min\{0, a_{ii_2 \dots i_m} : (i_2, \dots, i_m) \neq (i, \dots, i)\}, \quad (7)$$

$$\Delta_i(\mathcal{A}) = \sum \{(\beta_i(\mathcal{A}) - a_{ii_2 \dots i_m}) : (i_2, \dots, i_m) \neq (i, \dots, i)\}, \quad (8)$$

$$\Delta_j^i(\mathcal{A}) = \sum \{(\beta_j(\mathcal{A}) - a_{jj_2 \dots j_m}) : (j_2, \dots, j_m) \neq (i, \dots, i), (j_2, \dots, j_m) \neq (j, \dots, j)\}, \quad (9)$$

$$\Theta_i(\mathcal{A}) = \sum \{(a_{ii_2 \dots i_m} - \gamma_i(\mathcal{A})) : (i_2, \dots, i_m) \neq (i, \dots, i)\} \quad (10)$$

and

$$\Theta_j^i(\mathcal{A}) = \sum \{(a_{jj_2 \dots j_m} - \gamma_j(\mathcal{A})) : (j_2, \dots, j_m) \neq (i, \dots, i), (j_2, \dots, j_m) \neq (j, \dots, j)\}. \quad (11)$$

Next we recall the definitions of double B -tensor and quasi-double B -tensor defined by Li et al. in [3].

Definition 2.2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ with $a_{i \dots i} > \beta_i(\mathcal{A})$ for all i . Then \mathcal{A} is said to be a *double B -tensor* if :

- (a) for any $i \in \{1, \dots, n\}$, $a_{i \dots i} - \beta_i(\mathcal{A}) \geq \Delta_i(\mathcal{A})$,
- (b) for all $i, j \in \{1, \dots, n\}, i \neq j$, $(a_{i \dots i} - \beta_i(\mathcal{A}))(a_{j \dots j} - \beta_j(\mathcal{A})) > \Delta_i(\mathcal{A})\Delta_j(\mathcal{A})$.

Note that Chen et al. [2] also gave a definition of double B -tensor, which is different from the above one. A double B -tensor defined in [2] need not satisfy the condition (a) of definition 2.2. If we drop condition (a) from definition 2.2, then even order symmetric double B -tensors need not be positive definite[3]. Positive definiteness of double B -tensors plays a crucial role in locating the H -eigenvalues of even order symmetric tensors. So, in this paper, we follow the definition of double B -tensor as in [3].

Definition 2.3. [3] Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ with $a_{i \dots i} > \beta_i(\mathcal{A})$ for all i . Then \mathcal{A} is said to be a *quasi-double B -tensor* if :

$$(a_{i \dots i} - \beta_i(\mathcal{A}))(a_{j \dots j} - \beta_j(\mathcal{A}) - \Delta_j^i(\mathcal{A})) > (\beta_i(\mathcal{A}) - a_{ji \dots i})\Delta_j(\mathcal{A}).$$

Next we recall some definitions.

Definition 2.4. [3] Let \mathcal{A} be an m -order n -dimensional tensor. Then \mathcal{A} is called a *doubly strictly diagonally dominant tensor (DSDD)* if:

- (a) $|a_{i \dots i}| |a_{j \dots j}| > r_i(\mathcal{A}) r_j(\mathcal{A})$, for all $i, j, i \neq j$,
- (b) when $m > 2$, $|a_{i \dots i}| \geq r_i(\mathcal{A})$ for $i \in \{1, \dots, n\}$.

Definition 2.5. [3] Let \mathcal{A} be an m -order n -dimensional tensor. Then \mathcal{A} is called a *quasi-doubly strictly diagonally dominant tensor (Q-DSDD)* if:

$$|a_{i \dots i}| (|a_{j \dots j}| - r_j^i(\mathcal{A})) > r_i(\mathcal{A}) |a_{ji \dots i}|, \text{ for all } i, j, i \neq j.$$

A tensor \mathcal{A} is called a Z -tensor if there exists a tensor \mathcal{D} with nonnegative entries and a real number s such that $\mathcal{A} = s\mathcal{I} - \mathcal{D}$.

Li et al. [3] proved that the following important result using the fact that an even order symmetric double B -tensor (quasi-double B -tensor) can be decomposed into the sum of a doubly (quasi-doubly) strictly diagonally dominant symmetric Z -tensor and several positive multiples of partially all one tensors.

Theorem 2.1. [3] *The following statements are true:*

- (a) *All the H -eigenvalues of an even order symmetric double B -tensor are positive,*
- (b) *All the H -eigenvalues of an even order symmetric quasi-double B -tensor are positive.*

The following results are proved in [3]. We observed that these results contain minor mistakes. We give a counter example to these results and prove the correct versions (Theorem 3.1, Theorem 2.4).

Theorem 2.2. [3, Proposition 5] *Let \mathcal{A} be an m -order n -dimensional Z -tensor. Then:*

- (a) *\mathcal{A} is a double B -tensor if and only if \mathcal{A} is a DSDD tensor.*
- (b) *\mathcal{A} is a quasi-double B -tensor if and only if \mathcal{A} is a Q-DSDD tensor.*

Theorem 2.3. [3, Theorem 4] *Let \mathcal{A} be an m -order n -dimensional real symmetric tensor such that m is even. If \mathcal{A} is either a DSDD tensor or a Q-DSDD tensor, then \mathcal{A} is positive definite.*

The following example shows that Theorem 2.2 and 2.3 are not true. Consider the Z -matrix $A = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}$. Then A is both DSDD and Q-DSDD-matrix. But, it is neither a double B -matrix nor a quasi-double B -matrix. Also it is easy to see that A is not positive definite.

We will give a correct version of Theorem 2.2 in Section 3. Now we give a correct version of Theorem 2.3 as the follows.

Theorem 2.4. *Let \mathcal{A} be an m -order n -dimensional real symmetric tensor such that m is even and $a_{i\dots i} > 0$ for all $i \in [n]$.*

- (1) *If \mathcal{A} is a DSDD tensor, then \mathcal{A} is positive definite;*
- (2) *If \mathcal{A} is a Q-DSDD tensor, then \mathcal{A} is positive definite;*

Proof. The conclusion (1) follows directly from [5, Theorem 11]. If \mathcal{A} is a Q-DSDD tensor, then it is easy to see that there exist an index $i \in [n]$ such that $|a_{i\dots i}| > r_i(\mathcal{A})$. Now the conclusion (2) follows directly from [5, Theorem 13].

3 Double \overline{B} -tensors and quasi-double \overline{B} -tensors

In this section, first we introduce the notion of double \overline{B} -tensor and quasi-double \overline{B} -tensor.

Definition 3.1. Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be a tensor with the k th row tensor \mathcal{A}_k , $k \in [n]$ and $\overline{\mathcal{A}} = (\overline{a}_{i_1\dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be a tensor with the k th row tensor $\text{sign}(a_{k\dots k})\mathcal{A}_k$, $k \in [n]$. Then,

- (a) \mathcal{A} is called a double \overline{B} -tensor if $\overline{\mathcal{A}}$ is a double B -tensor,
- (b) \mathcal{A} is called a quasi-double \overline{B} -tensor if $\overline{\mathcal{A}}$ is a quasi-double B -tensor.

Note that, double \overline{B} -tensors discussed in this paper is different from double \overline{B} -tensors discussed in [2] because of the different definitions of double B -tensor.

Let \mathcal{A} be a double B -tensor. Since $a_{i\dots i} > \beta_i(\mathcal{A}) \geq 0$ for each $i \in [n]$, it is easy to see that $\overline{\mathcal{A}}$ is a double B -tensor, which means \mathcal{A} is a double \overline{B} -tensor. Hence,

$$\{\text{double } B\text{-tensors}\} \subset \{\text{double } \overline{B}\text{-tensors}\}.$$

Similarly, we have

$$\{\text{quasi-double } B\text{-tensors}\} \subset \{\text{quasi-double } \overline{B}\text{-tensors}\}.$$

Define

$$\delta_{i_1\dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Next we prove a correct version of Theorem 2.2.

Theorem 3.1. Let \mathcal{A} be an m -order n -dimensional tensor such that $\overline{\mathcal{A}}$ is a Z -tensor. Then

(a) \mathcal{A} is a double \overline{B} -tensor if and only if \mathcal{A} is a DSDD tensor.

(b) \mathcal{A} is a quasi-double \overline{B} -tensor if and only if \mathcal{A} is a Q-DSDD tensor.

Proof. (a) Let $\mathcal{B} = \overline{\mathcal{A}}$. Since \mathcal{B} is a Z -tensor, we have $\beta_i(\mathcal{B}) = 0$ and $r_i(\mathcal{B}) = \Delta_i(\mathcal{B}) = \sum_{\delta_{ii_2 \dots i_m}=0} -b_{ii_2 \dots i_m}$ for all i . Hence,

$$|b_{i \dots i}| |b_{j \dots j}| > r_i(\mathcal{B}) r_j(\mathcal{B})$$

if and only if

$$(b_{i \dots i} - \beta_i(\mathcal{B}))(b_{j \dots j} - \beta_j(\mathcal{B})) > \Delta_i(\mathcal{B}) \Delta_j(\mathcal{B}).$$

Also $b_{i \dots i} - \beta_i(\mathcal{B}) \geq \Delta_i(\mathcal{B})$ if and only if $|a_{i \dots i}| \geq r_i(\mathcal{A})$. Thus \mathcal{A} is a double \overline{B} -tensor if and only if \mathcal{A} is a DSDD tensor.

(b) Now, $\Delta_j^i(\mathcal{B}) = \sum_{\substack{\delta_{ij_2 \dots j_m}=0 \\ \delta_{jj_2 \dots j_m}=0}} (\beta_j(\mathcal{B}) - b_{jj_2 \dots j_m}) = r_j^i(\mathcal{B})$. Hence

$$b_{i \dots i} (b_{j \dots j} - r_j^i(\mathcal{B})) > r_i(\mathcal{B}) |b_{ji \dots i}|$$

if and only if

$$(b_{i \dots i} - \beta_i(\mathcal{B}))(b_{j \dots j} - \beta_j(\mathcal{B}) - \Delta_j^i(\mathcal{B})) > (\beta_j(\mathcal{B}) - b_{ji \dots i}) \Delta_i(\mathcal{B}).$$

Thus \mathcal{A} is a quasi double \overline{B} -tensor if and only if \mathcal{A} is a Q-DSDD tensor. \square

Let us recall the general product of two n -dimensional tensors defined by Shao in [10].

Definition 3.2. [10] Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{C}^n, m)$ and $\mathcal{B} = (b_{i_1 \dots i_k}) \in \mathcal{T}(\mathbb{C}^n, k)$. Define the product \mathcal{AB} to be the following tensor \mathcal{C} of order $(m-1)(k-1)+1$ and dimension n :

$$c_{i\alpha_1 \dots \alpha_{m-1}} = \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} b_{i_2 \alpha_1} \dots b_{i_m \alpha_{m-1}}, \quad (i \in [n], \alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1}).$$

By the definition of double \overline{B} -tensor and quasi-double \overline{B} -tensor, it is easy to prove the following.

Theorem 3.2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$. Then

(a) \mathcal{A} is a double \overline{B} -tensor if and only if there exists an $n \times n$ diagonal matrix D whose diagonal elements belong to the set $\{1, -1\}$ and a double B -tensor $\mathcal{B} \in \mathcal{T}(\mathbb{R}^n, m)$ such that $\mathcal{A} = D\mathcal{B}$,

(b) \mathcal{A} is a quasi-double \overline{B} -tensor if and only if there exists an $n \times n$ diagonal matrix D whose diagonal elements belong to the set $\{1, -1\}$ and a quasi-double B -tensor $\mathcal{B} \in \mathcal{T}(\mathbb{R}^n, m)$ such that $\mathcal{A} = D\mathcal{B}$.

Definition 3.3. Let $\mathcal{A} = (a_{i_1 \dots i_m})$ be an m -order n -dimensional tensor and $\alpha \subseteq \{1, \dots, n\}$ with $|\alpha| = r$. A *principal subtensor* $\mathcal{A}[\alpha]$ of the tensor \mathcal{A} with index set α is an m -order r -dimensional subtensor of \mathcal{A} consisting of r^m elements defined as follows:

$$\mathcal{A}[\alpha] = (a_{i_1 \dots i_m}) , \text{ where } i_1, \dots, i_m \in \alpha.$$

Theorem 3.3. Let \mathcal{A} be an m -order n -dimensional tensor.

- (a) If \mathcal{A} is a double \overline{B} -tensor, then $\mathcal{A}[\alpha]$ is a double \overline{B} -tensor for all $\alpha \subseteq \{1, \dots, n\}$.
- (b) If \mathcal{A} is a quasi-double \overline{B} -tensor, then $\mathcal{A}[\alpha]$ is a quasi-double \overline{B} -tensor for all $\alpha \subseteq \{1, \dots, n\}$.

Proof. (a) Let $\mathcal{B} = \overline{\mathcal{A}}$, $\alpha \subseteq \{1, \dots, n\}$ and $i \in \alpha$. Since $b_{i \dots i} > \beta_i(\mathcal{B})$, we have $b_{i \dots i} > \beta_i(\mathcal{B}[\alpha])$. Also, $b_{i \dots i} - \beta_i(\mathcal{B}) \geq \Delta_i(\mathcal{B})$ and $\Delta_i(\mathcal{B}) \geq \Delta_i(\mathcal{B}[\alpha])$. Thus $b_{i \dots i} - \beta_i(\mathcal{B}[\alpha]) \geq \Delta_i(\mathcal{B}[\alpha])$. Now,

$$\begin{aligned} (b_{i \dots i} - \beta_i(\mathcal{B}[\alpha]))(b_{j \dots j} - \beta_j(\mathcal{B}[\alpha])) &\geq (b_{i \dots i} - \beta_i(\mathcal{B}))(b_{j \dots j} - \beta_j(\mathcal{B})) \\ &> \Delta_i(\mathcal{B})\Delta_j(\mathcal{B}) \\ &\geq \Delta_i(\mathcal{B}[\alpha])\Delta_j(\mathcal{B}[\alpha]). \end{aligned}$$

Hence $\mathcal{A}[\alpha]$ is double \overline{B} -tensor.

(b) Let $i, j \in \alpha$ and $i \neq j$. We have $\Delta_j^i(\mathcal{B}) \geq \Delta_j^i(\mathcal{B}[\alpha])$. Now,

$$\begin{aligned} (b_{i \dots i} - \beta_i(\mathcal{B}[\alpha]))(b_{j \dots j} - \beta_j(\mathcal{B}[\alpha]) - \Delta_j^i(\mathcal{B}[\alpha])) &\geq (b_{i \dots i} - \beta_i(\mathcal{B}))(b_{j \dots j} - \beta_j(\mathcal{B}) - \Delta_j^i(\mathcal{B})) \\ &> (\beta_j(\mathcal{B}) - b_{ji \dots i})\Delta_i(\mathcal{B}) \\ &\geq (\beta_j(\mathcal{B}[\alpha]) - b_{ji \dots i})\Delta_i(\mathcal{B}[\alpha]). \end{aligned}$$

Thus $\mathcal{A}[\alpha]$ is a quasi-double \overline{B} -tensor. □

Definition 3.4. Let \mathcal{A} be an m -order n -dimensional tensor. Then $\mathcal{A}^+ = (a_{i_1 \dots i_m}^+)$ is an m -order n -dimensional tensor defined as $a_{i_1 \dots i_m}^+ = a_{i_1 \dots i_m} - \beta_{i_1}(\mathcal{A})$.

It is easy to see that \mathcal{A}^+ is a Z -tensor.

Theorem 3.4. Let \mathcal{A} be an m -order n -dimensional tensor. Then

- (a) \mathcal{A} is a double \overline{B} -tensor if and only if $\overline{\mathcal{A}}^+$ is a double B -tensor.
- (b) \mathcal{A} is a quasi-double \overline{B} -tensor if and only if $\overline{\mathcal{A}}^+$ is a quasi-double B -tensor.

Proof. Let $\mathcal{B} = \overline{\mathcal{A}}$ and $\mathcal{C} = \overline{\mathcal{A}}^+$.

(a) Suppose \mathcal{A} is a double \overline{B} -tensor. Then $c_{i \dots i} = b_{i \dots i} - \beta_i(\mathcal{B}) > 0$. Since \mathcal{C} is a Z -tensor and $\Delta_i(\mathcal{B}) = \Delta_i(\mathcal{C})$, we have $c_{i \dots i} \geq \Delta_i(\mathcal{C})$. Now, we have

$$(c_{i \dots i} - \beta_i(\mathcal{C}))(c_{j \dots j} - \beta_j(\mathcal{C})) = (b_{i \dots i} - \beta_i(\mathcal{B}))(b_{j \dots j} - \beta_j(\mathcal{B}))$$

and

$$\Delta_i(\mathcal{B})\Delta_j(\mathcal{B}) = \Delta_i(\mathcal{C})\Delta_j(\mathcal{C}).$$

Thus

$$(c_{i\dots i} - \beta_i(\mathcal{C}))(c_{j\dots j} - \beta_j(\mathcal{C})) > \Delta_i(\mathcal{C})\Delta_j(\mathcal{C})$$

if and only if

$$(b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B})) > \Delta_i(\mathcal{B})\Delta_j(\mathcal{B}).$$

Hence \mathcal{A} is a double \overline{B} -tensor if and only if $\overline{\mathcal{A}}^+$ is a double B -tensor.

(b) We have

$$(c_{i\dots i} - \beta_i(\mathcal{C}))(c_{j\dots j} - \beta_j(\mathcal{C}) - \Delta_j^i(\mathcal{C})) = (b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B}) - \Delta_j^i(\mathcal{B}))$$

and

$$(\beta_j(\mathcal{C}) - c_{ji\dots i})\Delta_j^i(\mathcal{C}) = (\beta_j(\mathcal{B}) - b_{ji\dots i})\Delta_j^i(\mathcal{B}).$$

Thus

$$(c_{i\dots i} - \beta_i(\mathcal{C}))(c_{j\dots j} - \beta_j(\mathcal{C}) - \Delta_j^i(\mathcal{C})) > (\beta_j(\mathcal{C}) - c_{ji\dots i})\Delta_j^i(\mathcal{C})$$

if and only if

$$(b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B}) - \Delta_j^i(\mathcal{B})) > (\beta_j(\mathcal{B}) - b_{ji\dots i})\Delta_j^i(\mathcal{B}).$$

Hence \mathcal{A} is a quasi-double \overline{B} -tensor if and only if $\overline{\mathcal{A}}^+$ is a quasi-double B -tensor. \square

Define

$$\alpha_i(\mathcal{A}) = \begin{cases} \beta_i(\mathcal{A}) & \text{if } a_{i\dots i} > 0, \\ \gamma_i(\mathcal{A}) & \text{if } a_{i\dots i} < 0, \end{cases}$$

where $\beta_i(\mathcal{A})$ and $\gamma_i(\mathcal{A})$ are defined in (6) and (7), respectively.

The following theorem gives an equivalent condition for a tensor to be a double \overline{B} -tensor.

Theorem 3.5. *Let \mathcal{A} be an m -order n -dimensional tensor. Then, \mathcal{A} is a double \overline{B} -tensor if and only if*

$$(a) \quad |a_{i\dots i}| > |\alpha_i(\mathcal{A})| \text{ for all } i,$$

$$(b) \quad |a_{i\dots i} - \alpha_i(\mathcal{A})| \geq \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|,$$

$$(c) \quad |a_{i\dots i} - \alpha_i(\mathcal{A})||a_{j\dots j} - \alpha_j(\mathcal{A})| > \left(\sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}| \right) \left(\sum_{\delta_{jj_2\dots j_m}=0} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}| \right).$$

Proof. Let $\mathcal{B} = \overline{\mathcal{A}}$. Then $b_{i\dots i} > \beta_i(\mathcal{B})$ if and only if $a_{i\dots i} > \beta_i(\mathcal{A})$ if $a_{i\dots i} > 0$ and $-a_{i\dots i} > -\gamma_i(\mathcal{A})$ if $a_{i\dots i} < 0$. Thus $b_{i\dots i} > \beta_i(\mathcal{B})$ if and only if $|a_{i\dots i}| > |\alpha_i(\mathcal{A})|$ for all i .

If $a_{i\dots i} < 0$, then

$$-a_{i\dots i} + \alpha_i(\mathcal{A}) = b_{i\dots i} - \beta_i(\mathcal{B}) \geq \sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}) = \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|$$

and if $a_{i\dots i} > 0$, then

$$a_{i\dots i} - \alpha_i(\mathcal{A}) = b_{i\dots i} - \beta_i(\mathcal{B}) \geq \sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}) = \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|.$$

Thus $|a_{i\dots i} - \alpha_i(\mathcal{A})| \geq \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|$ if and only if

$$b_{i\dots i} - \beta_i(\mathcal{B}) \geq \sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}).$$

Since

$$|a_{i\dots i} - \alpha_i(\mathcal{A})||a_{j\dots j} - \alpha_j(\mathcal{A})| = (b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B}))$$

and

$$\begin{aligned} & \left(\sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}) \right) \left(\sum_{\delta_{jj_2\dots j_m}=0} (\beta_j(\mathcal{B}) - b_{jj_2\dots j_m}) \right) \\ &= \left(\sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}| \right) \left(\sum_{\delta_{jj_2\dots j_m}=0} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}| \right), \end{aligned}$$

we have

$$|a_{i\dots i} - \alpha_i(\mathcal{A})||a_{j\dots j} - \alpha_j(\mathcal{A})| > \left(\sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}| \right) \left(\sum_{\delta_{jj_2\dots j_m}=0} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}| \right)$$

if and only if

$$(b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B})) > \left(\sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}) \right) \left(\sum_{\delta_{jj_2\dots j_m}=0} (\beta_j(\mathcal{B}) - b_{jj_2\dots j_m}) \right).$$

This proves the result. \square

The following result gives an equivalent condition for a tensor to be a quasi-double \overline{B} -tensor.

Theorem 3.6. *Let $\mathcal{A} = (a_{i_1\dots i_m})$ be an m -order n -dimensional tensor. Then \mathcal{A} is a quasi-double \overline{B} -tensor if and only if for each $i \in [n]$, $|a_{i\dots i}| > |\alpha_i(\mathcal{A})|$ and for any $i \neq j \in [n]$,*

$$\begin{aligned} & |a_{i\dots i} - \alpha_i(\mathcal{A})|(|a_{j\dots j} - \alpha_j(\mathcal{A})| - \sum_{\substack{\delta_{ij_2\dots j_m}=0 \\ \delta_{jj_2\dots j_m}=0}} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}|) \\ & > |\alpha_j(\mathcal{A}) - a_{ji\dots i}| \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|. \end{aligned} \tag{12}$$

Proof. Let $\mathcal{B} = \overline{\mathcal{A}}$. By the definition of $\alpha_i(\mathcal{A})$, one has that $b_{i\dots i} > \beta_i(\mathcal{A})$ if and only if $a_{i\dots i} > \beta_i(\mathcal{A})$ if $a_{i\dots i} > 0$ and $a_{i\dots i} < \gamma_i(\mathcal{A})$ if $a_{i\dots i} < 0$. Thus, we have $b_{i\dots i} > \beta_i(\mathcal{A})$ if and only if $|a_{i\dots i}| > |\alpha_i(\mathcal{A})|$.

Since,

$$\begin{aligned} & |a_{i\dots i} - \alpha_i(\mathcal{A})|(|a_{j\dots j} - \alpha_j(\mathcal{A})| - \sum_{\substack{\delta_{ij_2\dots j_m}=0 \\ \delta_{jj_2\dots j_m}=0}} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}|) \\ &= (b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B}) - \sum_{\substack{\delta_{ij_2\dots j_m}=0 \\ \delta_{jj_2\dots j_m}=0}} (\beta_j(\mathcal{B}) - b_{jj_2\dots j_m})) \end{aligned} \quad (13)$$

if and only if

$$\begin{aligned} & (\beta_j(\mathcal{B}) - b_{ji\dots i}) \sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}) \\ &= |\alpha_j(\mathcal{A}) - a_{ji\dots i}| \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}| \end{aligned} \quad (14)$$

Thus, we have

$$\begin{aligned} & |a_{i\dots i} - \alpha_i(\mathcal{A})|(|a_{j\dots j} - \alpha_j(\mathcal{A})| - \sum_{\substack{\delta_{ij_2\dots j_m}=0 \\ \delta_{jj_2\dots j_m}=0}} |\alpha_j(\mathcal{A}) - a_{jj_2\dots j_m}|) \\ &> |\alpha_j(\mathcal{A}) - a_{ji\dots i}| \sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A}) - a_{ii_2\dots i_m}|. \end{aligned} \quad (15)$$

if and only if

$$\begin{aligned} & (b_{i\dots i} - \beta_i(\mathcal{B}))(b_{j\dots j} - \beta_j(\mathcal{B}) - \sum_{\substack{\delta_{ij_2\dots j_m}=0 \\ \delta_{jj_2\dots j_m}=0}} (\beta_j(\mathcal{B}) - b_{jj_2\dots j_m})) \\ &> (\beta_j(\mathcal{B}) - b_{ji\dots i}) \sum_{\delta_{ii_2\dots i_m}=0} (\beta_i(\mathcal{B}) - b_{ii_2\dots i_m}). \end{aligned} \quad (16)$$

This proves the result. \square

4 New bounds for H -eigenvalues of even order symmetric tensors

In this section, we discuss results about location of the H -eigenvalues of even order symmetric tensors. First we state the following useful lemma. Proof is easy to verify.

Lemma 4.1. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$. Then 0 is an H -eigenvalue of \mathcal{A} if and only if 0 is an H -eigenvalue of $\overline{\mathcal{A}}$.*

In the following theorem we give a new region for the H -eigenvalues of even order symmetric tensors.

Theorem 4.1. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be an even order symmetric tensor and λ be an H -eigenvalue of \mathcal{A} . For any $i \in [n]$, Define*

$$\Lambda_i = [a_{i \dots i} - \beta_i(\mathcal{A}), a_{i \dots i} - \gamma_i(\mathcal{A})],$$

$$\tilde{\Lambda}_i = (a_{i \dots i} - \beta_i(\mathcal{A}) - \Delta_i(\mathcal{A}), a_{i \dots i} - \gamma_i(\mathcal{A}) - \Theta_i(\mathcal{A}))$$

and for any $i \neq j \in [n]$,

$$\Lambda_{ij}^1 = \{x \in (-\infty, \min\{a_{i \dots i}, a_{j \dots j}\}) : |a_{i \dots i} - \beta_i(\mathcal{A}) - x| |a_{j \dots j} - \beta_j(\mathcal{A}) - x| \leq \Delta_i(\mathcal{A}) \Delta_j(\mathcal{A})\},$$

$$\Lambda_{ij}^2 = \{x \in (a_{i \dots i}, a_{j \dots j}) : |a_{i \dots i} - \gamma_i(\mathcal{A}) - x| |a_{j \dots j} - \beta_j(\mathcal{A}) - x| \leq \Theta_i(\mathcal{A}) \Delta_j(\mathcal{A})\},$$

$$\Lambda_{ij}^3 = \{x \in (a_{j \dots j}, a_{i \dots i}) : |a_{i \dots i} - \beta_i(\mathcal{A}) - x| |a_{j \dots j} - \gamma_j(\mathcal{A}) - x| \leq \Delta_i(\mathcal{A}) \Theta_j(\mathcal{A})\},$$

$$\Lambda_{ij}^4 = \{x \in (\max\{a_{i \dots i}, a_{j \dots j}\}, +\infty) : |a_{i \dots i} - \gamma_i(\mathcal{A}) - x| |a_{j \dots j} - \gamma_j(\mathcal{A}) - x| \leq \Theta_i(\mathcal{A}) \Theta_j(\mathcal{A})\},$$

where $\beta_i(\mathcal{A})$ and $\gamma_i(\mathcal{A})$ are defined in (6) and (7). Let

$$\Lambda_{ij} = \begin{cases} \Lambda_{ij}^1 \cup \Lambda_{ij}^2 \cup \Lambda_{ij}^4, & \text{if } a_{i \dots i} \leq a_{j \dots j}, \\ \Lambda_{ij}^1 \cup \Lambda_{ij}^3 \cup \Lambda_{ij}^4, & \text{if } a_{i \dots i} \geq a_{j \dots j}. \end{cases}$$

Then,

$$\lambda \in \Lambda(\mathcal{A}) = \left(\bigcup_{i=1}^n \Lambda_i \right) \cup \left(\bigcup_{i=1}^n \tilde{\Lambda}_i \right) \cup \left(\bigcup_{\substack{i,j=1 \\ i \neq j}}^n \Lambda_{ij} \right). \quad (17)$$

Proof. Let λ be an H -eigenvalue of \mathcal{A} . Suppose that $\lambda \notin \Lambda(\mathcal{A})$. Observe that $\mathcal{A} - \lambda \mathcal{I}$ and \mathcal{A} have the same off-diagonal elements. Since $\lambda \notin \Lambda_i$, we have either $a_{i \dots i} - \lambda > \beta_i(\mathcal{A})$ or $a_{i \dots i} - \lambda < \gamma_i(\mathcal{A})$ for all $i \in [n]$. Thus $|a_{i \dots i} - \lambda| > |\alpha_i(\mathcal{A})| = |\alpha_i(\mathcal{A} - \lambda \mathcal{I})|$ for all $i \in [n]$. Since $\lambda \notin \tilde{\Lambda}_i$, we have either $(a_{i \dots i} - \beta_i(\mathcal{A}) - \Delta_i(\mathcal{A}) - \lambda) \geq 0$ or $(a_{i \dots i} - \gamma_i(\mathcal{A}) - \Theta_i(\mathcal{A}) - \lambda) \leq 0$. Thus $|a_{i \dots i} - \lambda - \alpha_i(\mathcal{A} - \lambda \mathcal{I})| \geq \sum_{\delta_{ii_2 \dots i_m} = 0} |\alpha_i(\mathcal{A} - \lambda \mathcal{I}) - a_{ii_2 \dots i_m}|$.

Let $i, j \in [n]$ and $i \neq j$. We may assume without loss of generality that $a_{i \dots i} \leq a_{j \dots j}$. If $\lambda \in (-\infty, a_{i \dots i})$, then $a_{i \dots i} - \lambda > 0$, $a_{j \dots j} - \lambda > 0$ and

$$|a_{i \dots i} - \beta_i(\mathcal{A}) - \lambda| |a_{j \dots j} - \beta_j(\mathcal{A}) - \lambda| > \Delta_i(\mathcal{A}) \Delta_j(\mathcal{A}).$$

If $\lambda \in (a_{i \dots i}, a_{j \dots j})$, then $a_{i \dots i} - \lambda < 0$, $a_{j \dots j} - \lambda > 0$ and

$$|a_{i \dots i} - \gamma_i(\mathcal{A}) - \lambda| |a_{j \dots j} - \beta_j(\mathcal{A}) - \lambda| > \Theta_i(\mathcal{A}) \Delta_j(\mathcal{A}).$$

If $\lambda \in (a_{j\dots j}, \infty)$, then $a_{i\dots i} - \lambda < 0$, $a_{j\dots j} - \lambda < 0$ and

$$|a_{i\dots i} - \gamma_i(\mathcal{A}) - \lambda| |a_{j\dots j} - \gamma_j(\mathcal{A}) - \lambda| > \Theta_i(\mathcal{A})\Theta_j(\mathcal{A}).$$

From the above observations, we have the following inequality

$$\begin{aligned} & |a_{i\dots i} - \lambda - \alpha_i(\mathcal{A} - \lambda\mathcal{I})| |a_{j\dots j} - \lambda - \alpha_j(\mathcal{A} - \lambda\mathcal{I})| \\ & > \left(\sum_{\delta_{ii_2\dots i_m}=0} |\alpha_i(\mathcal{A} - \lambda\mathcal{I}) - a_{ii_2\dots i_m}| \right) \left(\sum_{\delta_{ji_2\dots i_m}=0} |\alpha_j(\mathcal{A} - \lambda\mathcal{I}) - a_{ji_2\dots i_m}| \right). \end{aligned}$$

Thus, by Theorem 3.5, we have $\mathcal{A} - \lambda\mathcal{I}$ is a double \overline{B} -tensor. By Theorem 2.1 (a) and Lemma 4.1, 0 is not an H -eigenvalue of $\mathcal{A} - \lambda\mathcal{I}$. Thus there does not exist a nonzero vector $x \in \mathbb{R}^n$ such that $(\mathcal{A} - \lambda\mathcal{I})x^{m-1} = 0$, which implies λ is not an H -eigenvalue of \mathcal{A} . This is a contradiction. Therefore, $\lambda \in \Lambda(\mathcal{A})$. \square

The above theorem is an analogous version of [7, Theorem 3.3] for tensors. The following theorem is a refinement of the above theorem. The eigenvalue region we get in this theorem are smaller than the above one.

Theorem 4.2. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be an even order symmetric tensor and λ be an H -eigenvalue of \mathcal{A} . Define*

$$\Psi_i = [a_{i\dots i} - \beta_i(\mathcal{A}), a_{i\dots i} - \gamma_i(\mathcal{A})], \quad i \in [n]$$

and for any $i \neq j \in [n]$,

$$\begin{aligned} \Psi_{ij}^1 &= \{x \in (-\infty, \min\{a_{i\dots i}, a_{j\dots j}\}) : |a_{i\dots i} - \beta_i(\mathcal{A}) - x| (|a_{j\dots j} - \beta_j(\mathcal{A}) - x| - \Delta_j^i(\mathcal{A})) \\ &\leq (\beta_j(\mathcal{A}) - a_{ji\dots i})\Delta_i(\mathcal{A})\}, \end{aligned}$$

$$\Psi_{ij}^2 = \{x \in (a_{i\dots i}, a_{j\dots j}) : |a_{i\dots i} - \gamma_i(\mathcal{A}) - x| (|a_{j\dots j} - \beta_j(\mathcal{A}) - x| - \Delta_j^i(\mathcal{A})) \leq (\beta_j(\mathcal{A}) - a_{ji\dots i})\Theta_i(\mathcal{A})\},$$

$$\Psi_{ij}^3 = \{x \in (a_{j\dots j}, a_{i\dots i}) : |a_{i\dots i} - \beta_i(\mathcal{A}) - x| (|a_{j\dots j} - \gamma_j(\mathcal{A}) - x| - \Theta_j^i(\mathcal{A})) \leq (a_{ji\dots i} - \gamma_j(\mathcal{A}))\Delta_i(\mathcal{A})\},$$

$$\begin{aligned} \Psi_{ij}^4 &= \{x \in (\max\{a_{i\dots i}, a_{j\dots j}\}, +\infty) : |a_{i\dots i} - \gamma_i(\mathcal{A}) - x| (|a_{j\dots j} - \gamma_j(\mathcal{A}) - x| - \Theta_j^i(\mathcal{A})) \\ &\leq (a_{ji\dots i} - \gamma_j(\mathcal{A}))\Theta_i(\mathcal{A})\}, \end{aligned}$$

where $\beta_i(\mathcal{A})$ and $\gamma_i(\mathcal{A})$ are defined in (6) and (7). Let

$$\Psi_{ij} = \begin{cases} \Psi_{ij}^1 \cup \Psi_{ij}^2 \cup \Psi_{ij}^4, & \text{if } a_{i\dots i} \leq a_{j\dots j}, \\ \Psi_{ij}^1 \cup \Psi_{ij}^3 \cup \Psi_{ij}^4, & \text{if } a_{i\dots i} \geq a_{j\dots j}. \end{cases}$$

Then,

$$\lambda \in \Psi(\mathcal{A}) = \left(\bigcup_{i=1}^n \Psi_i \right) \cup \left(\bigcup_{\substack{i,j=1 \\ i \neq j}}^n \Psi_{ij} \right). \quad (18)$$

Proof. Let λ be an H -eigenvalue of \mathcal{A} . Suppose that $\lambda \notin \Psi(\mathcal{A})$. Then for any $i \in [n]$, $|a_{i\dots i} - \lambda| > |\alpha_i(\mathcal{A} - \lambda\mathcal{I})|$. Let $i \neq j \in [n]$. Without loss of generality, assume $a_{i\dots i} \leq a_{j\dots j}$. If $\lambda \in (-\infty, a_{i\dots i})$, then

$$|a_{i\dots i} - \beta_i(\mathcal{A}) - \lambda|(|a_{j\dots j} - \beta_j(\mathcal{A}) - \lambda| - \Delta_j^i(\mathcal{A})) > (\beta_j(\mathcal{A}) - a_{ji\dots i})\Delta_i(\mathcal{A}).$$

If $\lambda \in (a_{i\dots i}, a_{j\dots j})$, then

$$|a_{i\dots i} - \gamma_i(\mathcal{A}) - \lambda|(|a_{j\dots j} - \beta_j(\mathcal{A}) - \lambda| - \Delta_j^i(\mathcal{A})) > (\beta_j(\mathcal{A}) - a_{ji\dots i})\Theta_i(\mathcal{A}).$$

If $\lambda \in (a_{j\dots j}, \infty)$, then

$$|a_{i\dots i} - \gamma_i(\mathcal{A}) - \lambda|(|a_{j\dots j} - \gamma_j(\mathcal{A}) - \lambda| - \Theta_j^i(\mathcal{A})) > (a_{ji\dots i} - \gamma_j(\mathcal{A}))\Theta_i(\mathcal{A}).$$

Thus in all three cases we have,

$$\begin{aligned} & |a_{i\dots i} - \lambda - \alpha_i(\mathcal{A} - \lambda\mathcal{I})|(|a_{j\dots j} - \lambda - \alpha_j(\mathcal{A} - \lambda\mathcal{I})| - \sum_{\substack{\delta_{ij2\dots jm}=0 \\ \delta_{jj2\dots jm}=0}} |\alpha_j(\mathcal{A} - \lambda\mathcal{I}) - a_{jj2\dots jm}|) \\ & > |\alpha_j(\mathcal{A} - \lambda\mathcal{I}) - a_{ji\dots i}| \sum_{\delta_{ii2\dots im}=0} |\alpha_i(\mathcal{A} - \lambda\mathcal{I}) - a_{ii2\dots im}|. \end{aligned}$$

By Theorem 3.6, we have $\mathcal{A} - \lambda\mathcal{I}$ is a quasi-double \overline{B} -tensor. Also, by Theorem 2.1 (b) and Lemma 4.1, 0 is not an H -eigenvalue of $\mathcal{A} - \lambda\mathcal{I}$. Thus there does not exist a nonzero vector $x \in \mathbb{R}^n$ such that $(\mathcal{A} - \lambda\mathcal{I})x^{m-1} = 0$, which means λ is not an H -eigenvalue of \mathcal{A} . This is a contradiction. Therefore, $\lambda \in \Psi(\mathcal{A})$. \square

We call $\Lambda(\mathcal{A})$ and $\Psi(\mathcal{A})$ the double \overline{B} -intervals and quasi-double \overline{B} -intervals of \mathcal{A} , respectively. Observe that the double \overline{B} -intervals of \mathcal{A} and quasi-double \overline{B} -intervals have a nature similar to the Brauer-type eigenvalues inclusion set $\Omega(\mathcal{A})$ stated in Theorem 1.3. Now, a question is nature, when range is better for determining the location of the H -eigenvalues of \mathcal{A} . We will discuss this interesting question in the following section.

5 Comparisons between Brauer-type eigenvalues inclusion set, double \overline{B} -intervals and quasi-double \overline{B} -intervals

We begin this section with the following observation. Li et al., [3, Proposition 4] proved that if \mathcal{A} is a double B -tensor, then \mathcal{A} is a quasi-double B -tensor. Thus we have, if \mathcal{A} is a double \overline{B} -tensor, then \mathcal{A} is a quasi-double \overline{B} -tensor. In the next theorem, we prove the quasi-double \overline{B} -intervals is contained in the double \overline{B} -intervals.

Theorem 5.1. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be an even order symmetric tensor and λ be an H -eigenvalue of \mathcal{A} . Then,*

$$\Psi(\mathcal{A}) \subseteq \Lambda(\mathcal{A}), \tag{19}$$

where $\Lambda(\mathcal{A})$ and $\Psi(\mathcal{A})$ are defined in (17) and (18), respectively.

Proof. Let $\lambda \notin \Lambda(\mathcal{A})$. Then, by Theorem 3.5, $\mathcal{A} - \lambda\mathcal{I}$ is a double \overline{B} -tensor. By the above observation, $\mathcal{A} - \lambda\mathcal{I}$ is a quasi-double \overline{B} -tensor. Thus, by Theorem 3.6, we have $\lambda \notin \Psi(\mathcal{A})$. Hence $\Psi(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$. \square

The following example shows that $\Psi(\mathcal{A})$ and Brauer-type eigenvalues inclusion set are not comparable.

Example 5.1. Consider the symmetric tensors $\mathcal{A}_1 = (a_{i_1 i_2 i_3 i_4})$ and $\mathcal{A}_2 = (b_{i_1 i_2 i_3 i_4})$ of order 4 dimension 2 defined as follows:

$$\begin{aligned} a_{1111} &= 18, \quad a_{2222} = 20, \quad a_{1222} = a_{2122} = a_{2212} = a_{2221} = 3, \\ a_{1122} &= a_{2211} = a_{1221} = a_{2112} = a_{2121} = a_{1212} = 2, \\ a_{1112} &= a_{2111} = a_{1211} = a_{1121} = 2, \end{aligned}$$

and

$$\begin{aligned} b_{1111} &= 2, \quad b_{2222} = 6, \quad b_{1222} = b_{2122} = b_{2212} = b_{2221} = 4, \\ b_{1122} &= b_{2211} = b_{1221} = b_{2112} = b_{2121} = b_{1212} = -2, \\ b_{1112} &= b_{2111} = b_{1211} = b_{1121} = 5. \end{aligned}$$

By using the methods of Theorem 1.3 and Theorem 4.2, one can get the location of H -eigenvalues of \mathcal{A}_1 and \mathcal{A}_2 listed in the following Table 1.

Table 1: Comparisons of Brauer-type eigenvalues inclusion set and quasi-double \overline{B} -intervals

	\mathcal{A}_1	\mathcal{A}_2
Brauer-type eigenvalues inclusion set	[3, 36.6119]	[-22.2560, 28.6844]
quasi-double \overline{B} -intervals	[9, 36.6119]	[-24.9257, 32.6068]
H -eigenvalues	15, 35.1469	-20.2289, 16.0666

As we can see in the table above, the quasi-double \overline{B} -intervals is better when estimating the location of H -eigenvalues of \mathcal{A}_1 while the Brauer-type eigenvalues inclusion set is more precise for locating the H -eigenvalues of \mathcal{A}_2 .

So, there is no indication that $\Psi(\mathcal{A})$ is tighter than $\Omega(\mathcal{A})$ or $\Omega(\mathcal{A})$ is tighter than $\Psi(\mathcal{A})$. They supplement each other. \square

Now, we construct a more precise region by using the results of Theorem 1.3 and Theorem 4.2.

Theorem 5.2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathcal{T}(\mathbb{R}^n, m)$ be an even order symmetric tensor and $\Omega(\mathcal{A})$, $\Psi(\mathcal{A})$ be defined in (5), (18), respectively. If λ is an H -eigenvalue of \mathcal{A} , then

$$\lambda \in \Upsilon(\mathcal{A}) = \Omega(\mathcal{A}) \cap \Psi(\mathcal{A}). \quad (20)$$

6 Concluding remarks

We introduced two new classes of tensors called double \overline{B} -tensors and quasi-double \overline{B} -tensors and established some of their properties. Using the properties, we derived two regions $\Lambda(\mathcal{A})$, $\Psi(\mathcal{A})$ called double \overline{B} -intervals, quasi-double \overline{B} -intervals, respectively. We observed that quasi-double \overline{B} -intervals are smaller than double \overline{B} -intervals. We discussed the relationship between Brauer-type eigenvalues inclusion set $\Omega(\mathcal{A})$ and the quasi-double \overline{B} -intervals $\Psi(\mathcal{A})$ and constructed the region $\Upsilon(\mathcal{A})$, which is more precise for the location of the H -eigenvalues of even order symmetric tensors. We investigated the location of H -eigenvalues of even order symmetric tensors. Do the double \overline{B} -intervals $\Lambda(\mathcal{A})$ and the quasi-double \overline{B} -intervals $\Psi(\mathcal{A})$ hold for even order nonsymmetric tensors and odd order tensors? In the future, we will research these problems.

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